circular orifice at relatively low p_a/p_t (Fig. 2) indicate a value for C_D of about 0.87. The measurements by Perry,[16] at a somewhat higher p_a/p_t, indicate a slightly lower value for C_D. The predictions shown[14,17] agree reasonably well with the measurements.

It appears that the measured values of C_D for choked nozzles with comparatively small r_c/r_{th} and relatively large σ operated at relatively low p_a/p_t as they usually are, tend toward the sharp-edged orifice value as indicated in Fig. 2 by the curve faired through the data. However in the limiting case $r_c/r_{th} \to 0$, the measurements by Thornock[18] indicate a dependence of C_D on σ, with smaller σ's leading to higher C_D's (Fig. 2). This trend is in agreement with earlier measurements of Ref. 19 at larger p_a/p_t and predictions,[20] and does imply a dependence of C_D on σ for nozzles with very small r_c/r_{th}. For this situation, C_D might be estimated from the dashed curves in Fig. 2 for the σ's indicated. Little information is available on C_D for nozzles with relatively small σ and small r_c/r_{th}. It would appear that values of C_D for such nozzles would be higher.

Concluding Remarks

For supersonic nozzles with ratios of throat radius of curvature to throat radius between 0 and 2.0, tested at relatively high Reynolds numbers, the reduction in mass flow rate below the ideal one-dimensional flow value was caused primarily by the throat configuration rather than by boundary-layer effects. From the collection of data for air flows, the flow coefficient can be estimated (essentially the inviscid flow coefficient, $C_{D_{\rm inv}}$) from the curves representing the data (Fig. 2). Consequently, information is available on how much the throat might be enlarged to accommodate the otherwise reduced mass flow rate or how much the chamber pressure might be increased. At lower Reynolds numbers where viscous (boundary-layer) effects become important, the actual flow coefficient might be calculated from the following relation that was derived in Ref. 1 by considering axisymmetric flow in the throat plane‡

$$C_D = C_{D_{\rm inv}} - 2[(\overline{\rho u})_{\rm inv}/(\rho u)^*](\delta^*/r_{th})[1 - (\delta^*/2r_{th})]$$

where $(\overline{\rho u}]_{\rm inv}$ is the average value of the mass flux across the displacement thickness. To a first approximation, $(\overline{\rho u})_{\rm inv}$ might be evaluated at the wall for an inviscid flow.

References

1 Massier, P. F., "Viscous Effects on the Flow Coefficient for a Supersonic Nozzle," *AIAA Journal*, Vol. 8, No. 3, March 1970, pp. 605–607.

2 Cuffel, R. F., Back, L. H., and Massier, P. F., "Transonic Flow Field in a Supersonic Nozzle with Small Throat Radius of Curvature," *AIAA Journal*, Vol. 7, No. 7, July 1969, pp. 1364–1366.

3 Durham, F. P., "The Performance Characteristics of Small Rocket-Type Nozzle," Summary Rept. 6, Contract N-123S-82337, T.O. 1, June 1955, Univ. of Colorado, Boulder, Colo.

4 Strand, L., Newton, J. F., Jr., and Herrera, J. G., "Initial Performance of a New Nozzle Gas-Dynamics Test Facility," Tech. Memo 33-310, Nov. 1966, Jet Propulsion Lab., Pasadena, Calif.

5 Norton, D. and Shelton, S., "Performance of Rocket Nozzles with Low Radius of Curvature Ratios," *JPL Space Programs Summary No. 37-55*, Vol. III, Feb. 1969, Pasadena, Calif., pp. 167–169.

6 Back, L. H., Massier, P. F., and Gier, H. L., "Comparison of Measured and Predicted Flows through Conical Supersonic Nozzles, with Emphasis on the Transonic Region," *AIAA Journal*, Vol. 3, No. 9, Sept. 1965, pp. 1606–1614.

7 Sauer, R., "General Characteristics of the Flow Through Nozzles at Near Critical Speeds," TM-1147, June 1947, NACA.

8 Oswatitsch, K. and Rothstein, W., "Flow Pattern in a Converging-Diverging Nozzle," TM-1215, March 1949, NACA.

‡ For two-dimensional flow through a nozzle with a rectangular cross section, $C_D = C_{D_{\rm inv}} - [(\overline{\rho u})_{\rm inv}/(\rho u)^*](\delta^*/h_{th})$; where h_{th} is the throat half-height.

9 Hall, I. M., "Transonic Flow in Two-Dimensional and Axially-Symmetric Nozzles," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. XV, Pt. 4, 1962, pp. 487–508.

10 Kleigel, J. R. and Levine, J. N., "Transonic Flow in Small Throat Radius of Curvature Nozzles," *AIAA Journal*, Vol. 7, No. 7, July 1969, pp. 1375–1378.

11 Prozan, R. J., "Transonic Flow in a Converging-Diverging Nozzle," LMSC/HREC D162177, Contract NAS7-743, 1970, Lockheed Missiles & Space Co., Huntsville, Ala.

12 Milne-Thomson, *Theoretical Hydrodynamics*, Macmillan, New York, 1968, p. 310.

13 Bloch, E., "Numerical Solution of Free Boundary Problems by the Method of Steepest Descent," *The Physics of Fluids Supplement II, High Speed Computing in Fluid Dynamics*, Dec. 1969, pp. 129–132.

14 Benson, R. S. and Pool, D. E., "The Compressible Flow Discharge Coefficients for a Two-Dimensional Slit," *International Journal of Mechanical Sciences*, Vol. 7, May 1965, pp. 337–353.

15 Durham, F. P., "The Performance Characteristics of Small Rocket-Type Nozzles," Rept. 2, Contract N-123S-82337, T.O. 1, June 1953, Univ. of Colorado, Boulder, Colo.

16 Perry, J. A., Jr., "Critical Flow Through Sharp-Edged Orifices," *Transactions of ASME*, Vol. 71, 1949, pp. 757–764.

17 Jobson, D. A., "On the Flow of a Compressible Fluid through Orifices," *Proceedings Institution of Mechanical Engineers*, Vol. 169, 1955, pp. 767–772.

18 Thornock, R. L., "Experimental Investigation of the Flow Through Convergent Conical Nozzles," Document D6-20375, Sept. 1968, The Boeing Co., Seattle, Wash.

19 Grey, R. E. and Wilsted, H. D., "Performance of Conical Jet Nozzles in Terms of Flow and Velocity Coefficients," TN-1757, Nov. 1948, NACA.

20 Wehofer, S. and Moger, W. C., "Transonic Flow in Conical Convergent and Convergent-Divergent Nozzles with Non-uniform Inlet Conditions," AIAA Paper 70-635, San Diego, Calif., 1970.

Nonlinear Proportional Navigation and the Minimum Time-to-Turn

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Nomenclature

A	= azimuth angle of the velocity vector
$B, \ldots, E$	= coefficient functions in the quartic Eq. (23)
g	= acceleration due to gravity
k	= Ng, missile lateral acceleration (N is a signed number)
LOS	= line of sight between missile and target
R	= missile-target separation range along the LOS
t,T	= time and its reciprocal ($T = t^{-1}$), respectively
V	= flight speed
x,y	= inertially-fixed Cartesian coordinate axes in the plane
$\alpha,\beta,\gamma,\epsilon$	= coefficients in the biquadratic Eq. (29)
Δ,δ	= function differences in Eqs. (11–12, 18–21)
ΔHE	= missile heading error at $t = 0$
σ	= LOS angle measured from the x-axis (Fig. 1)
τ	= square of reciprocal time T in Eq. (33), ($\tau = T^2 = (t^{-2})$

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Subscripts

M,T = missile and target, respectively
t = minimum time-to-turn until $\dot{\sigma} = 0$
0 = conditions at time $t = 0$ at the beginning of terminal maneuver

Introduction

SIMULATION and postflight analyses indicate that a homing missile ostensibly guided by proportional navigation (PN) generates a two-part flight path. The missile first pulls a nearly circular turn until the LOS angular rate $d\sigma/dt \equiv \dot{\sigma} \rightarrow 0$. Thereafter it flies straight, maintaining a constant bearing with the nonaccelerating target until interception (Fig. 1). The minimum time-to-turn t_t for the first phase is usually obtained through various computer programs.

This Note develops a quadratic equation whose larger positive root closely approximates t_t^{-1}, obviating the need for computer solution (either by simulation or calculation) of the nonlinear kinematic differential equations.

Tactical Situation and Terminal Guidance

Point mass missile and target move in the (x,y) plane at constant speeds V_M and V_T, respectively, with $R(t)$ the distance between them. The target's rectilinear path, the missile V_M vector and the LOS make the angles A_T (fixed), $A_M(t)$ and $\sigma(t)$, respectively, with the x axis. During homing, PN guidance generates lateral steering commands $V_M\dot{A}_M$ by requiring $\dot{A}_M$ to be proportional to $\dot{\sigma}$. The missile turns to drive $\dot{\sigma}(t) \rightarrow 0$ in order to reduce the heading error inherited from the externally-guided midcourse phase.

Linearized analytic PN homing theory[1] predicts $\dot{\sigma} \rightarrow 0$ *as* $R(t) \rightarrow 0$. However, the implied missile flight-path behavior has not been verified in practice or simulation. The discrepancy is attributable to nonlinearities, time lags and other imperfections in the hardware (real or simulated) but omitted from the theory. The large initial $\dot{\sigma}$ saturates the guidance system, and time lags delay immediate response to subsequent smaller signals. Consequently $\dot{\sigma} \rightarrow 0$ *before* $R(t)$ does.

This dual nonlinear/linear mode, originally unintentional, forms the basis of one nonlinear PN guidance scheme,[2] in which only the algebraic sign of $\dot{\sigma}(t)$ is used to command full lateral acceleration until $\dot{\sigma} \cong 0$.

Minimum Time-to-Turn

The missile possesses lateral maneuver capability Ng, so that the turning rate is

$$\dot{A}_M = Ng/V_M \equiv k \tag{1}$$

and the heading angle is

$$A_M(t) = A_{M0} + kt \tag{2}$$

The missile velocity components are

$$\dot{x}_M(t) = V_M\mathrm{c}A_M(t) \qquad \dot{y}_M(t) = V_M\mathrm{s}A_M(t) \tag{3}$$

where c ≡ cosine and s ≡ sine henceforth. The position coordinates, measured from the axes origin are

$$x_M(t) = V_M\int_0^t \mathrm{c}A_M(u)du = [\mathrm{s}(A_{M0} + kt) - \mathrm{s}A_{M0}]V_M/k \tag{4}$$

$$y_M(t) = V_M\int_0^t \mathrm{s}A_M(u)du = [-\mathrm{c}(A_{M0} + kt) + \mathrm{c}A_{M0}]V_M/k \tag{5}$$

The corresponding target position and velocity components are

$$x_T(t) = x_{T_0} + tV_T\mathrm{c}A_T \qquad y_T(t) = y_{T_0} + tV_T\mathrm{s}A_T \tag{6}$$

$$\dot{x}_T = V_T\mathrm{c}A_T \qquad \dot{y}_T = V_T\mathrm{s}A_T \tag{7}$$

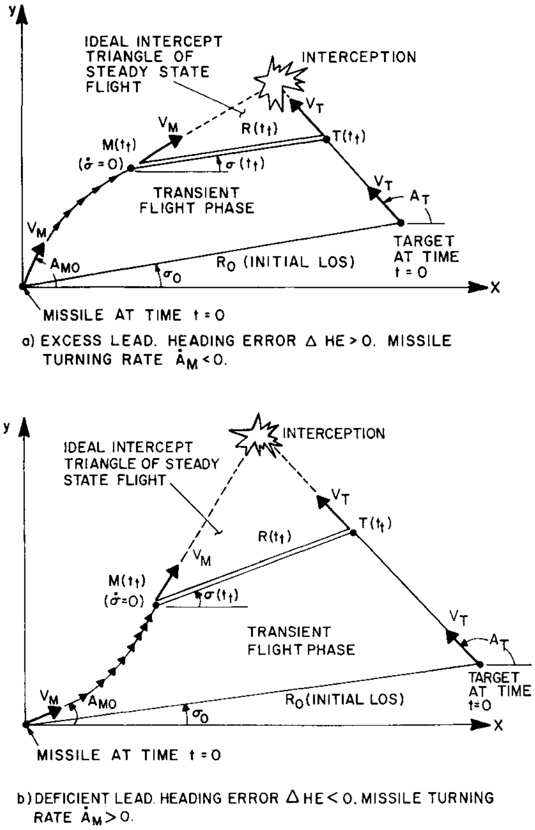

Fig. 1 Homing tactical situation.

where the target's initial position (Fig. 1) is

$$x_{T_0} = R_0\mathrm{c}\sigma_0 \qquad y_{T_0} = R_0\mathrm{s}\sigma_0 \tag{8}$$

Using the basic definition of the LOS angle $\sigma(t)$,

$$\tan\sigma(t) = \Delta y/\Delta x \tag{9}$$

we obtain

$$\dot{\sigma}(t) = (\Delta x\Delta\dot{y} - \Delta y\Delta\dot{x})/R^2 \tag{10}$$

where

$$\Delta x = x_T(t) - x_M(t) \qquad \Delta y = y_T(t) - y_M(t) \tag{11}$$

$$\Delta\dot{x} = \dot{x}_T - \dot{x}_M(t) \qquad \Delta\dot{y} = \dot{y}_T - \dot{y}_M(t) \tag{12}$$

$$R(t) = [(\Delta x)^2 + (\Delta y)^2]^{1/2} \tag{13}$$

For later convenience, $\dot{R}(t)$ may be represented in either form below:

$$\dot{R}(t) = (\Delta x\Delta\dot{x} + \Delta y\Delta\dot{y})/R = V_T\mathrm{c}[A_T - \sigma(t)] - V_M\mathrm{c}[A_M(t) - \sigma(t)] \tag{14}$$

Now the condition $\dot{\sigma}(t_t) = 0$ implies that the turning time t_t can be calculated by equating the numerator of Eq. (10) to zero and obtaining

$$\Delta x/\Delta\dot{x} = \Delta y/\Delta\dot{y} \tag{15}$$

Substituting Eqs. (11) and (12) and rearranging, we obtain a trigonometric equation whose smallest positive root, if it exists, will be t_t. (The initial range R_0 and the maximum allowable $\dot{A}_M$ may not permit $\dot{\sigma} = 0$ to be attained with $\dot{R}(t_t) < 0$). If an acceptable t_t results, then at that instant $\dot{\sigma}(t_t) = 0$, and the missile heading angle, missile and target positions, LOS angle, range and range rate are calculated from Eqs. (2, 4–6, 9, 13, and 14), respectively. These quantities define the new initial conditions for the start of the second part of the flight path, and are needed to predict the corresponding terminal values[3] at the point of closest separation in the case of unsuccessful interception.

Approximate Time-to-Turn

In the past, a convenient but sometimes invalid approximation to t_t was obtained from the quotient

$$t_t \cong |\Delta \text{HE}/\dot{A}_M| \tag{16}$$

Here $\Delta\text{HE} = A_{M_0}{}^* - A_M{}^*$ is the initial heading error of the missile at $t = 0$, with $A_M{}^*$ calculated from the law of sines applied to the closed ideal interception triangle, i.e.,

$$\text{s}(A_M{}^* - \sigma_0) = (V_T/V_M)\,\text{s}(A_T - \sigma_0) \tag{17}$$

Equation (16) contains errors due to the unrealistic assumption of instantaneous $\dot{A}_M$ (implying zero turning radius), the lack of distinction between positive and negative heading errors despite asymmetrical target-crossing geometry (see Fig. 1), and by the time-varying nature of the intercept triangle itself that changes shape and location during the turning interval t_t.

A better approximation can be developed in the following manner: expand the trigonometric functions contained in Eqs. (11) and (12) in a power series about $t = 0$, retaining terms with powers of t through second order. (First-order approximations to the trigonometric terms lead to excessive error in calculating the turning time. Second-order terms permit successful treatment of at least $|\Delta\text{HE}| \leq 10°$.) We obtain

$$\Delta x \cong x_{T_0} + t\delta\dot{x}_0 - Jt^2/2 \qquad \Delta y \cong y_{T_0} + t\delta\dot{y}_0 - Ht^2/2 \tag{18}$$

$$\Delta\dot{x} \cong \delta\dot{x}_0 + Jt + Hkt^2/2 \qquad \Delta\dot{y} \cong \delta\dot{y}_0 - Ht + Jkt^2/2 \tag{19}$$

wherein

$$\delta\dot{x}_0 = V_T\text{c}A_T - V_M\text{c}A_{M0} \qquad H = Ng\text{c}A_{M0} \tag{20}$$

$$\delta\dot{y}_0 = V_T\text{s}A_T - V_M\text{s}A_{M0} \qquad J = Ng\text{s}A_{M0} \tag{21}$$

Substitution of Eqs. (18) and (19) into Eq. (15) leads to an equation whose solution will yield the approximate time-to-turn,

$$\frac{x_{T_0} + t\delta\dot{x}_0 - Jt^2/2}{\delta\dot{x}_0 + Jt + Hkt^2/2} = \frac{y_{T_0} + t\delta\dot{y}_0 - Ht^2/2}{\delta\dot{y}_0 - Ht + Jkt^2/2} \tag{22}$$

Cross-multiplying Eq. (22), rearranging the resulting quartic polynomial in order of descending powers of t with the leading coefficient positive, introducing the reciprocal transformation $t = T^{-1}$ and normalizing the coefficient of T^4 to unity, we obtain the basic quartic equation,

$$T^4 + BT^3 + CT^2 + DT + E = 0 \tag{23}$$

The coefficients $B, \ldots, E$ are defined by combinations of functions of the known initial conditions involving the geometry and kinematics, becoming signed constants when evaluated at $t = 0$:

$$B = -Ng\text{c}(A_{M0} - \sigma_0)/(R_0\dot{\sigma}_0) \tag{24}$$

$$C = \frac{(Ng)^2\text{s}(A_{M0} - \sigma_0)}{2V_M(R_0\dot{\sigma}_0)} - \frac{Ng[V_T\text{c}(A_T - A_{M0}) - V_M]}{2R_0(R_0\dot{\sigma}_0)} \tag{25}$$

$$D = -(Ng)^2V_T\text{s}(A_T - A_{M0})/2V_MR_0(R_0\dot{\sigma}_0) \tag{26}$$

$$E = (Ng)^3/4V_MR_0(R_0\dot{\sigma}_0) \tag{27}$$

$$(R_0\dot{\sigma}_0) = V_T\text{s}(A_T - \sigma_0) - V_M\text{s}(A_{M0} - \sigma_0) \tag{28}$$

As a consequence of the transformation $t = T^{-1}$, we now seek the larger positive root of Eq. (23), an advantage in numerical work.

The existence of this root (and hence the feasibility of the interception) may be ascertained by Descartes' Rule of Signs[4] applied to Eq. (23), i.e., the number of positive real roots of a polynomial equation never exceeds the number of coefficient sign changes and if less, by an even number. This yields some immediate helpful criteria. No positive roots exist if the signs are all positive. Alternating signs imply the roots are all positive. One sign change implies one positive root. If $E < 0$, there is at least one positive root as the quartic is an even function. (To obtain additional criteria, change T to $-T$, reapply Descartes' Rule, and subtract the number of negative roots found from four.) Henceforth, assume at least one positive root exists.

Equation (23) may be solved by approximating the quartic with a quadratic equation as follows: assume it is factorable into the product of two quadratics

$$(T^2 + \alpha T + \beta)(T^2 + \gamma T + \epsilon) = 0 \tag{29}$$

and by coefficient comparison with Eq. (23)

$$\alpha + \gamma = B \quad \beta\gamma + \alpha\epsilon = D$$
$$\beta + \epsilon + \alpha\gamma = C \quad \beta\epsilon = E \tag{30}$$

The general technique[5] of solving Eq. (30) assumes an α and evaluates γ, β, and ϵ from the first three equations, with the last equation serving as a check on the α value assumed. However, as a result of the transformation ($t = T^{-1}$), R_0 appears as an explicit factor in the D and E coefficient denominators, rendering them relatively small in magnitude compared to B and C. Hence, γ and ϵ are also small compared with α and β; so to a first approximation,[6]

$$\alpha \cong B \quad \beta \cong C \quad \gamma \cong (DC - BE)/C^2 \quad \epsilon \cong E/C \tag{31}$$

The corresponding quadratic equation approximating Eq. (23) is then

$$T^2 + BT + C = 0 \tag{32}$$

and is readily solvable for the larger positive root t_t^{-1}. (If γ and ϵ were not sufficiently small, second- and higher-order approximations can be obtained through explicit, formal iterative relations).[6]

The condition $|B|, |C| \gg |D|, |E|$ depends on adequate root separation. If the roots are too close, an analytic application of the Graeffe root-squaring process[4] will transform Eq. (23) into another quartic whose roots are the squares of those of Eq. (23), thereby increasing the root separation prior to factoring by Eq. (29). The corresponding transformed quadratic (in terms of the original coefficients) that approxi-

Table 1 Initial conditions and results

Case	N	A_{M0}	$\sim\Delta$HE	Target position, ft x_{T_0}	Target position, ft y_{T_0}	Turning time, sec Eq. (32)	Turning time, sec Eq. (33)
1) Excess lead	−5	55°01′36″	9°50′	76340.03419	21307.43801	3.46	3.50
2) Deficit lead	+5	34°58′23″	−9°40′	77731.08054	19916.39080	3.51	3.47

mates Eq. (23) is

$$\tau^2 + (2C - B^2)\tau + (C^2 - 2BD + 2E) = 0 \qquad (33)$$

and the larger positive root τ is then t_t^{-2}. (If necessary the numerical version of the Graeffe process[4] could be applied repetitively to Eq. (23) until the roots are sufficiently separated.)

Numerical Results

Two numerical examples (both with known exact solutions $t_t = 3.50$ sec and $A_M(t_t) = 45°$) are presented. Initial conditions common to both are $V_M = 3220'/\text{s}$, $V_T = 2^{-1/2}V_M$, $A_T = 150°$. The remaining initial conditions and results are given in Table 1.

The last tabular entry requires explanation. whereas Eq. (32) gave $t_t = 3.51$ sec, the presumably more accurate Eq. (33) gave 3.47 sec. Investigation disclosed that in the case of deficient lead, the condition $\dot{\sigma}(t_t) = 0$ is relatively insensitive to small changes in time, so that truncation and round-off errors in computing the coefficients Eqs. (24–28) will perturb the root of Eq. (33).

In conclusion, either Eq. (32) or (33), if appropriate, may be used to approximate the reciprocal of the minimum time-to-turn t_t required by a constant speed missile (with constrained lateral acceleration) to achieve a constant-bearing homing course.

References

[1] Murtagh, S. A. and Criel, H. E., "Fundamentals of Proportional Navigation," *IEEE Spectrum*, Vol. 3, No. 12, Dec. 1966, pp. 75–85.

[2] Meyer, J. A. and Bland, J. G., "Minimum Information Terminal Guidance," *Proceedings National Electronics Conference*, Vol. 22, 1966, pp. 703–708.

[3] Rawling, A. G., "Prediction of Terminal Variables in Homing," *Journal of Spacecraft and Rockets*, Vol. 7, No. 6, June 1970, pp. 764–766.

[4] Uspensky, J. V., *Theory of Equations*, 1st ed., McGraw-Hill, New York, 1948, pp. 121–124 and 318–319.

[5] Chestnut, H. and Mayer, R. W., *Servomechanisms and Regulating Systems Design*, Vol. 1, Wiley, New York, 1951, pp. 131–133. (The authors attribute the technique to A. Porter.)

[6] Milne-Thomson, L. M., *Theoretical Aerodynamics*, 2nd ed.. St. Martin's Press, New York, pp. 336–337.